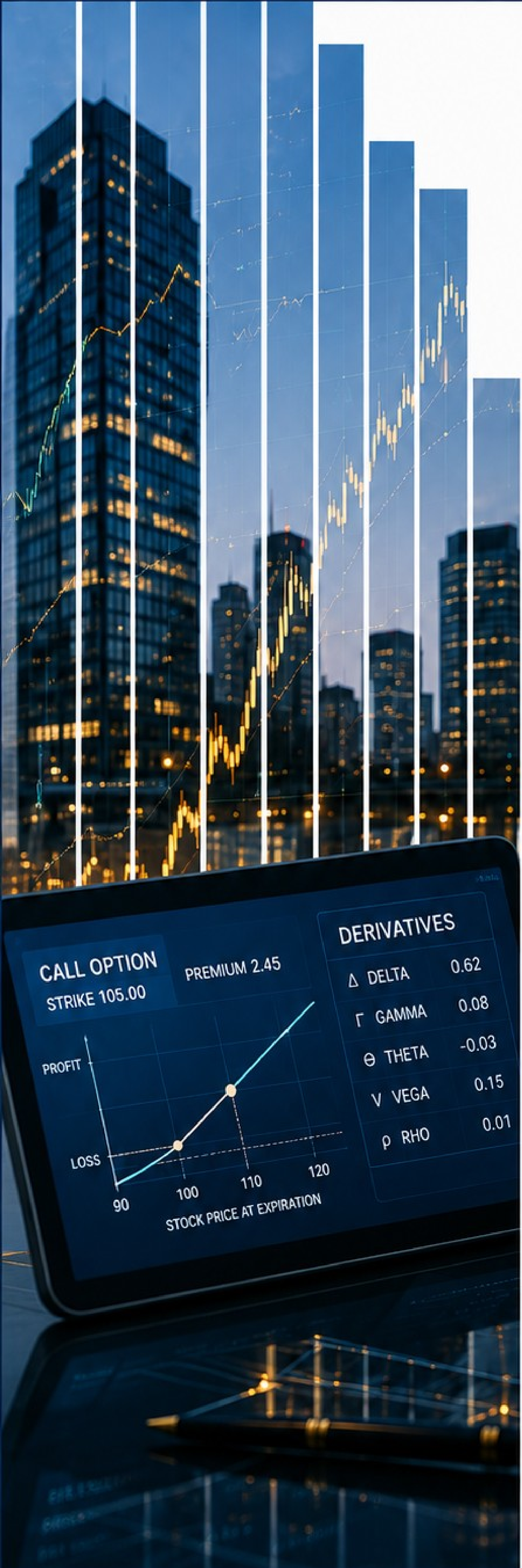

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THE CAPITAL ASSET PRICING THEORY AND ITS MISCONCEPTIONS

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ABSTRACT

The CAPM is the fundamental model for pricing financial securities. Nevertheless, the way it is proved in Finance textbooks can be fairly confusing, and more complicated than necessary; with an excessive use of figures at the expense of equations. In addition, depending on the Finance textbook, the set of assumptions that are supposedly needed to prove the CAPM may actually differ. This paper tries to provide an intuitive and straightforward proof of the CAPM and it also illustrates at which instances of the proof the traditional assumptions are actually needed. The main conclusion is that a much simpler version of the CAPM would possibly be as formidable and ingenious as the traditional one and yet, with a much more intuitive proof and fewer unrealistic assumptions.

Key Words: Capital Asset Pricing Model, Market Portfolio, Risk Free Security

JEL codes: G10, G11, G12, G14

INTRODUCTION

The Capital Pricing Model of Lintner (1965) and Sharpe (1965), or CAPM, is the fundamental model for pricing financial securities. Nevertheless, it is far from being an obvious model and the way it is proved in Finance textbooks[1] can, in many circumstances, be fairly confusing, more complicated than necessary and with an excessive use of charts and figures at the expense of equations – which makes a proper understanding of the model still more difficult.

In addition, depending on the Finance textbook, the set of assumptions that are supposedly needed to prove the CAPM may actually differ. For instance, a typical set of assumptions are the following: (i) there are many atomistic investors who are price takers, (ii) all investors are mean-variance optimizers, (iii) investments are made over the course of one single period (iv) investments are limited to the set of publicly traded securities (v) all investors analyze securities in the same manner and have the same economic view of the world (vi) all investors can borrow or lend without restrictions at the Risk Free Rate (vii) all information is available simultaneously to all investors.

The objective of this paper is to provide a straightforward proof of the CAPM, illustrate which of the above assumptions are actually needed for the proof and, if so, at exactly what point of the proof they are to be used. The paper has four major conclusions:

(1) The first one is that the famous CAPM result that the excess return of a given security is proportional – or in CAPM terminology, a "beta" – of the excess return of the Portfolio, does not depend on any of the assumptions above. It is, in fact, derived from the application of a simple rule of three in a model with three linear equations. That is, by measuring two variables – such as, for instance, the return of a given security and the return of the Portfolio - in terms of a third variable – such as, for instance, the Risk Free Rate – and then comparing these two results with each other, the conclusion will always be that one of the results will be a "beta" of the other. And this is independent of whether we are measuring returns, as is the case of CAPM, or, for instance, rainy days. The only thing that indeed changes is the formula for the "beta", which does depend on the exact definitions of each of these variables. .

(2) The second conclusion is that the return of a given security as well as the return of the Portfolio can actually be measured with respect to any other security in the Portfolio - not necessarily the Risk Free Rate. As mentioned above, the only thing that indeed changes is the formula for the "beta" that, in this specific case, will also need to account for the covariance of this "Non-Risk-Free" security with the Portfolio.

(3) The third conclusion is that it is possible to derive an analogous expression to that of the CAPM even for the case in which the investor´s Portfolio is not the Market Portfolio. Furthermore, even in this case, it would still be possible to measure the "beta" of security i, in terms of the investor´s Portfolio as well as in terms of the Market Portfolio – sufficing for that a small adjustment in the formula for the "beta".

(4) The fourth conclusion is that the CAPM basic structure is completely derived from a partial equilibrium framework – actually from the first order conditions of one and only representative investor. Thus, the assumptions of multiple investors, homogeneous expectations and the whole apparatus related to the market structure and the supply of securities – in other words general equilibrium – are only needed to guarantee the very unrealistic conclusion that all investors hold the Market Portfolio. What´s more: all these additional and unrealistic assumptions will result in only a minor modification in the CAPM formula for the "beta".

The paper is organized as follows. Section (2), defines the investor´s maximization problem and derives the first order conditions for a maximum. These conditions turn out to be linear equations, independently of how the variables inside the Utility Function are defined. Section (3) then illustrates

[1] See for instance Sharpe (1970), Elton and Grueber (2014) and Bodie, Merton and Cleeton (2009)

that, by simply subtracting two of the variables in terms of a third one and then measuring these "excesses" in terms of each other; this procedure will always result in a "beta relationship". Section (4) then defines the variables inside the representative investor Utility Function as the Expected Return and the Variance of the Portfolio. It then derives an expression for the "beta" for the case in which there is no Risk Free Security. In Section (5) the Market Portfolio is priced in terms of the investor´s Portfolio. In addition, a new "beta" expression is derived for the case in which one still wants to price a security in terms of the Market Portfolio - even if the investor´s Portfolio is not the Market Portfolio. Section (6) then assumes the existence of a Risk Free Security and derives the new equation for "beta" but still in terms of the investor´s (not the Market) Portfolio. Section (7) assumes the traditional general equilibrium framework of many atomistic investors that are all identical to the representative investor and also have homogeneous expectations, in order to derive the "beta" expression for the case in which the investor´s Portfolio is also the Market Portfolio. This is the conventional CAPM formula as derived and explained in Finance textbooks. Finally, section (8) concludes.

THE REPRESENTATIVE INVESTOR PROBLEM

The representative investor problem is to determine a value for a_i, i = 1, 2,.....,N so as to maximize a Utility Function $U[E[R_p];\sigma_p^2]$ defined over $E[R_p]$ and σ_p^2, and subject to the constraint that $k = \sum_{i=1}^{N} a_i$. In Section (4) bellow we will define the variables a_i, $E[R_p]$ and σ_p^2 as being, respectively, the share of security i in Portfolio P, the Expected Return of Portfolio P and the Variance of Portfolio P – with N being the number of securities in the Portfolio and k the leverage of the Portfolio.

The objective of this section, however, is to derive a few traditional results of the Capital Asset Pricing Model that do not depend on these definitions, but solely on the linear structure of the first-order conditions.

The Lagrangian for the representative investor problem is given by:

1) $$L = U[E[R_p];\sigma_p^2] + \lambda(k - \sum_{i=1}^{N} a_i)$$

Where λ is the Lagrange multiplier.

The optimal solution for each a_i, *i = 1, 2,.....,N* is determined by the following system of *N* first order equations subject to the constraint that the sum of all a_i, *i = 1, 2,.....,N* is equal to *k*:

2) $$U_1 \frac{\partial E[R_p]}{\partial a_i} + U_2 \frac{\partial \sigma_p^2}{\partial a_i} = \lambda, \qquad for\ \ i = 1, 2, ..., N$$

Where U_1 and U_2 are, correspondingly, the marginal utilities with respect to $E[R_p]$ and σ_p^2.

That is, the representative investor will be altering each a_i, *i = 1, 2, ...,N* to the point at which his/her utility gain, $U_1 \frac{\partial E[R_p]}{\partial a_i} + U_2 \frac{\partial \sigma_p^2}{\partial a_i}$, will be exactly the same for all a_i, *i = 1, 2,.....,N*, and equal to the Lagrange multiplier *λ*.

THE LINEAR STRUCTURE AND THE "BETA"

Notice that the system of N first order conditions described in 2) also applies for $i = F$, which we will soon – but not yet - define to be the Risk Free Security. Thus, for $i = F$:

3) $$U_1 \frac{\partial E[R_p]}{\partial a_F} + U_2 \frac{\partial \sigma_p^2}{\partial a_F} = \lambda,$$

Substituting for the Lagrange multiplier λ in the system of N equations in 2) we can rewrite the first order conditions in terms of "excesses" with respect to F in the following manner:

4) $$U_1 \left(\frac{\partial E[R_p]}{\partial a_i} - \frac{\partial E[R_p]}{\partial a_F} \right) = -U_2 \left(\frac{\partial \sigma_p^2}{\partial a_i} - \frac{\partial \sigma_p^2}{\partial a_F} \right)$$ for $i = 1, 2, \ldots\ldots N;$

And equation 4) above also applies to security P, which we will soon – but not yet - define to be the representative agent own Portfolio. That is:

5) $$U_1 \left(\frac{\partial E[R_{thp}]}{\partial a_p} - \frac{\partial E[R_p]}{\partial a_F} \right) = -U_2 \left(\frac{\partial \sigma_p^2}{\partial a_p} - \frac{\partial \sigma_p^2}{\partial a_F} \right)$$

Dividing 4) by 5), canceling out the marginal utilities U_1 e U_2 and rearranging terms, we get to the following linear expression relating the "excess of the derivative of $E[R_p]$ with respect to a_i, over the derivative of $E[R_p]$ with respect to a_F" to the "excess of the derivative of $E[R_p]$ with respect to a_P, over the derivative of $E[R_p]$ with respect to a_F". That is:

6) $$\left(\frac{\partial E[R_p]}{\partial a_i} - \frac{\partial E[R_p]}{\partial a_F} \right) = \beta_i \left(\frac{\partial E[R_p]}{\partial a_p} - \frac{\partial E[R_p]}{\partial a_F} \right)$$

And the expression. for β_i, which will later in the paper become the "CAPM beta", is given by:

7) $$\beta_i = \frac{\left(\frac{\partial \sigma_p^2}{\partial a_i} - \frac{\partial \sigma_p^2}{\partial a_F} \right)}{\left(\frac{\partial \sigma_p^2}{\partial a_p} - \frac{\partial \sigma_p^2}{\partial a_F} \right)}$$

Two things to notice about equation 6); both of them being independent of how we define $E[R_p]$ and σ_p^2. Firstly, notice that "the excess of the derivative of $E[R_p]$ with respect to a_i, over the derivative of $E[R_p]$ with respect to a_F" is linear on the "excess of the derivative of $E[R_p]$ with respect to a_P, over the derivative of $E[R_p]$ with respect to a_F ", with the proportionality constant being precisely the "beta" term, β_i. Next sections, when we define $E[R_p]$ as the Expected Return of the Portfolio, security F as the Risk Free Security and security P as being the representative investor Portfolio, this result will become one of the traditional conclusions of the Capital Asset Pricing Model: namely that the Expected Excess Return of security i over the Risk Free Rate will be proportional to the Expected Excess

Return of the Portfolio with respect to the Risk Free Rate – with the proportionality constant being given by the "beta" of security *i*, β_i.

The second thing to notice is that equations 6) and 7) do not depend on the parameters of the representative investor Utility Function. In particular, the "beta" relation between the two "Expected Excess Returns" is independent of the tastes or the risk aversion of the representative investor. What happens is that – since the "excess of *i* over *F*", as well as "the excess of *P* over *F*" are both functions of the marginal utility of the representative investor – this marginal utility ends up cancelling out when one measures one "excess" in terms of the other.

Notice that both of these results depend solely on the simple rule of three that we applied to the first order conditions in 2): firstly using the "excess of *i* over *F*" to substitute for the Lagrange multiplier λ, and then measuring this "excess of *i* over *F*"" in terms of the "excess of *P* over *F*"" to cancel out the marginal utilities. Therefore, both of these results are completely independent on the specifications of the variables $E[R_p]$ and σ_p^2 that will be carried out in the next sections.

DEFINING THE VARIABLES IN THE UTILITY FUNCTION

Let´s now define the variables $E[R_p]\,;\,\sigma_p^2$, as being, respectively, the Expected Return and the Variance of a Portfolio *P* composed of *N* securities; with each security representing a share a_i,*i = 1, 2, ...,N* of the Portfolio.

From basic statistics, the Expected Return of a Portfolio *P* with *N* securities would be given by:

8) $$E[R_p] = \sum_{i=1}^{N} a_i E[r_i]$$

Where $E[r_i]$ is the Expected Return of security *i* and a_i is the share of security *i* in the Portfolio *P*, for *i = 1, ..., N.*

The Variance of the Portfolio would be given by:

9) $$\sigma_p^2 = \sum_{i=1}^{N} a_i^2 \sigma_i^2 + \sum_{i=1}^{N} a_i \sum_{\substack{j=1 \\ j\neq i}}^{N} a_j \sigma_{i,j}$$

Where σ_i^2 is the Variance of security *i, i = 1, ..., N* and $\sigma_{i,j}$ is the covariance of security *i* with security *j*, for *i =1, ..., N* and *j =1, ..., N* and *j ≠ i .*

We can now use the above expressions for the Expected Return and the Variance of the Portfolio to calculate the derivatives with respect to the shares a_i , *i = 1, ..., N*. We have that:

10) $$\frac{\partial E[R_p]}{\partial a_i} = E[r_i]$$

11) $$\frac{\partial \sigma_p^2}{\partial a_i} = 2a_i\sigma_i^2 + 2\sum_{\substack{j=1 \\ j\neq i}}^{N} a_j \sigma_{i,j}$$

for *i =1, 2 ,.......N;*

Notice that the derivatives in 10) and 11) also apply for security F as well as the Portfolio P. In addition, the expression $\sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^{N} a_j \sigma_{i,j}$ in equation 11) is precisely the covariance of security i with the Portfolio P, for $i = 1, 2,....N$. In addition, for the specific case in which security i is the Portfolio P, then the covariance of Portfolio P with itself would actually be the variance of the Portfolio σ_p^2. That is:

12) $$\frac{\partial \sigma_P^2}{\partial a_i} = 2\sigma_{i,P}$$, for $i = 1, 2,.....N$

13) $$\frac{\partial \sigma_P^2}{\partial a_p} = 2\sigma_P^2$$, for $i = P$

Substituting into equations 6) and 7) we have that:

14) $$E[r_i - r_F] = \beta_i E[r_p - r_F]$$

And equation 14) is the traditional Capital Asset Pricing Model expression relating the Expected Excess Return of security i with respect to security F as a "beta" of the Expected Excess Return of the Portfolio P with respect to security F.

The expression for the "beta" of security i is given by:

15) $$\beta_i = \frac{(\sigma_{i,P} - \sigma_{F,P})}{(\sigma_P^2 - \sigma_{F,P})}$$

An important thing to notice about equations 14) and 15) is that, so far in the analysis, we have not assumed the existence of a Risk Free Security. In this sense, the linear, or "beta" relation, between the Excess Return of security i and the Excess Return of the Portfolio P, can be defined in terms of any security in the Portfolio - and not just in terms of the Risk Free Security. As a matter of fact, the only difference between these two cases would be in the "beta" equation. In the absence of a Risk Free Security the expression for the "beta" would be that in equation 15) above; while, in the presence of a Risk Free Security, there would be a few minor changes – as will be illustrated in section 6) below.

USING THE PORTFOLIO TO PRICE THE MARKET PORTFOLIO

.For the specific case in which security i is the Market Portfolio M, then equation 15) above would give the "beta" of the Market Portfolio in terms of the representative investor own Portfolio. That is:

16) $$\beta_M = \frac{(\sigma_{MP} - \sigma_{F,P})}{(\sigma_P^2 - \sigma_{F,P})}$$

And notice that, for the case in which the representative investor´s Portfolio P is exactly equal to the Market Portfolio M – which only happens under very specific hypothesis as will be discussed in section 7) – then β_M would be equal to 1.

In addition, from equations 14), 15) e 16), we could also price a given security i in terms of the Market Portfolio M – even for the case in which the representative investor does not hold the Market Portfolio. That is:

17) $$E[r_i - r_F] = \left(\frac{\beta_i}{\beta_M}\right) E[r_M - r_F]$$

In other words,, even for the case in which the representative investor´s Portfolio P is not the Market Portfolio; and yet he/she still wants to price the Expected Excess Return of a given security i in terms of the Market Portfolio M – then the "beta" of security i with respect to the Market Portfolio would be precisely the "beta" of security *i* with respect to the investor´s own Portfolio *P*, divided by the "beta" of the Market Portfolio *M* also with respect to the investor´s own Portfolio *P*.

Substituting for equations 15) and 16) and cancelling out the common denominators, we would arrive at the following expression for the Expected Excess Return of security *i*, with respect to the Market Portfolio *M* – for the case in which the investor´s Portfolio is not the Market Portfolio:

18) $$E[r_i - r_F] = \frac{\sigma_{MP} - \sigma_{F,P}}{\sigma_{MP} - \sigma_{F,P}}\, E[r_M - r_F]$$

DEFINING THE RISK FREE RATE

Once we define security *F* as the Risk Free Security, the equation for the "beta" of security *i* as defined in 15) acquires its more conventional form. That is, if *F* is Risk Free, it has zero volatility and, in addition, its covariance with the Portfolio *P*, $\sigma_{F,P}$, is also equal to zero. Substituting into 15) yields:

19) $$\beta_i = \frac{\sigma_{i,P}}{\sigma_P^2}$$

Notice that the expression in 19) is already very close to the traditional expression for the "CAPM beta"; namely that "beta" is a ratio between the covariance of security *i* with the Market Portfolio and the variance of the Market Portfolio. The only thing that is still missing is to show under what conditions the investor´s Portfolio *P* is actually the Market Portfolio *M*. And to reach this sole – and unrealistic – conclusion, we will need the great majority of those seven assumptions listed in the introduction of this paper.

GENERAL EQUILIBRIUM

To get to the conclusion that the representative investor´s Portfolio *P* is exactly the Market Portfolio, we need to assume that all other investors solve exactly the same problem as the representative investor. In other words, all other investors need to be "Mean-Variance optimizers" and need to share the exactly same views and information on the Variance-Covariance Matrix and Expected Returns of the securities in the market. Under these circumstances, equations 14) and 19) would then become the traditional CAPM equations:

20) $$E[r_i - r_F] = \beta_i E[r_M - r_F]$$

21) $$\beta_i = \frac{\sigma_{i,M}}{\sigma_M^2}$$

Notice that the great majority of the assumptions listed in the Introduction of this paper – in particular those that refer to the many atomistic investors with the same economic view of the world, homogeneous expectations and with simultaneous access and equal information – are only needed to derive the conclusion that the investor´s Portfolio is equal the Market Portfolio. These are, in my opinion, very unrealistic assumptions and are simply not necessary to derive the major CAPM results.

In fact, if the objective were just to measure the contribution of a given security in terms of the investor´s own Portfolio, as it is done in equations 14) and 15), none of these extra assumptions would be necessary. In addition, as shown in Section 5), even in the case in which the investor´s Portfolio is not the Market Portfolio, it would still be possible to measure the contribution of a given security in terms of the Market Portfolio. And this would be done simply by making a minor modification in the equation for the "CAPM beta", as illustrated in equations 16) and 17) – without the need of the numerous unrealistic assumptions required to force every single investor to hold the Market Portfolio.

CONCLUSION

The major conclusion of the *Capital Asset Pricing Model* is that the Expected Excess Return of a given security with respect to the Risk Free Rate, is a proportion “beta” of the Expected Excess Return of the Market Portfolio with respect to this same Risk Free Rate. With such simple, and yet not obvious, conclusion, the CAPM revolutionized the practice of Finance and became the fundamental model to price financial securities.

Nevertheless, the proof of the CAPM - at least in the way it is usually done in Finance textbooks - tend to be more complicated and less transparent than strictly needed; with an excessive use of figures at the expense of equations. This paper tries to fill this gap by providing an intuitive and straightforward proof of the CAPM with the sole use of equations and without resorting to efficient frontiers, security market lines or capital market lines figures.

In addition, this paper also aims to illustrate at which instances in the proof of the CAPM, the traditional assumptions are actually used. In fact, we show that the ”beta” relation between the Expected Excess Returns of two different securities is just independent of all the traditional assumptions of the CAPM; depending only on a simple rule of three. Notwithstanding, the CAPM assumptions will matter for the exact formula of the “beta” – which have a few minor modifications depending on whether there is a Risk Free Security or not, or on whether one wants to measure the contribution of a given security in terms of the investor´s Portfolio or in terms of the Market Portfolio.

As a matter of fact, the great majority of the traditional CAPM assumptions – and notably the most unrealistic ones – are needed solely to show that the investor´s Portfolio is actually the Market Portfolio. And to reach that conclusion, one needs to assume that all investors are exactly the same which, obviously, would result in all investors holding the same exact Portfolio which would then be – almost by definition – the Market Portfolio.

A much simpler version of the CAPM, in which the “Expected Excess Return of a security in relation to a certain other security” would be a proportion “beta” of the “Expected Excess Return of the investor´s Portfolio in relation to that same other security” – a version that does neither define a Risk Free Security nor that the investor´s Portfolio is the Market Portfolio – would possibly be as formidable and ingenious as the traditional version of the CAPM. And yet, it would be much more intuitive and with a remarkably simpler proof than the traditional version – as this paper tried to illustrate.

In other words, the CAPM most important conclusions require only two assumptions; namely that the investor be “Mean-Variance optimizer” and that investments are made of the course of one single period. These are the only assumptions needed to guarantee the linear structure of the first order conditions and thus the proportionality or “beta” relation between any two securities in the Portfolio. In this sense, the most important conclusions of the CAPM do not need the existence of a Risk Free Security nor the unrealistic conclusion that the representative investor holds the Market Portfolio.

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