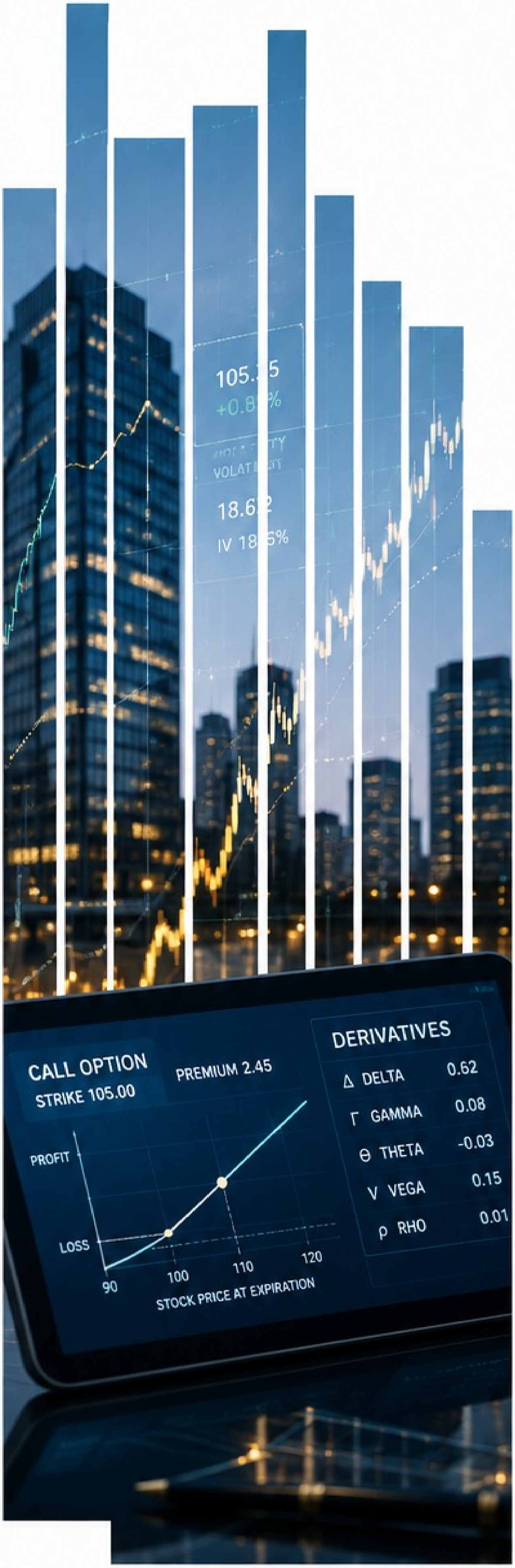

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Revisiting Modern Portfolio Theory and Its Misinterpretations

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REVISITING MODERN PORTFOLIO THEORY AND ITS MISINTERPRETATIONS

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ABSTRACT

This paper revisits Modern Portfolio Theory and derives eleven properties of Efficient Allocations and Efficient Portfolios in order to analyze two misinterpretations of the Theory: the Mutual Fund Theorem and the Roles of Diversification in an Efficient Portfolio. A mean-variance efficient allocation is one in which the totality of the investment is allocated in the Minimum Variance Portfolio, whilst the Risk Portfolio comprises exclusively of shorts and longs that exactly cancel each other out. That is, no net resources are ever allocated in the Risk Portfolio and a change in investor´s risk-return preferences will leave the allocation between the Minimum Variance and Risk Portfolios completely unaltered - only changing the magnitudes of the shorts and longs within the Risk Portfolio. This interpretation of the Mutual Fund theorem contrasts with the traditional interpretation that a change in risk-return preferences will actually shift resources from the Minimum Variance to the Risk Portfolio. Secondly, in an Efficient Portfolio, Diversification acquires another important role. In addition to the traditional role of minimizing variance, Diversification is also an instrument to efficiently increase risk and the magnitudes of the shorts and longs within the Risk Portfolio.

I. Introduction

The main conclusions of Modern Portfolio Theory have been extensively discussed in the literature and are widely known and used in the practice of Asset Management. Concepts such as systemic and idiosyncratic risks, the idea that not all risks command expected returns and the benefits of diversification are now common knowledge among academics as well as Investment professionals. Yet, before Markowitz (1952) and Sharpe (1965) contributions, they were not. During that dark age of Finance, concentrated portfolios and naïve diversification were the norm, while Portfolio Construction was much more a learn-by-doing activity than the science it turned out to be.

Yet, despite Modern Portfolio Theory´s important contributions, two major misinterpretations still persist. They have to do with the Mutual Fund Theorem and the true role of Diversification within a Portfolio. Part of the fault lies with traditional Finance textbooks that, with the intention of over-simplifying Modern Portfolio Theory for the greater audience, have resorted to the extensive use of charts and simplistic examples, at the expense of the more complex, and yet more precise, matrix algebra.

This paper tries to clarify such misinterpretations and, in the spirit of Merton (1972), revisits Modern Portfolio Theory with the sole use of matrix algebra. The paper also allows for different degrees of Portfolio leverage (read different efficient frontiers), and derive eleven properties of Efficient Allocations that are barely mentioned in Finance textbooks. These properties bring new intuition to the traditional Mutual Fund Theorem by reinterpreting the Minimum Variance and Risk Portfolios as, in practitioners' jargon, Strategic and Tactical Portfolios. They also bring new intuition to the roles of Diversification: in an Efficient Portfolio, Diversification is also an important instrument to efficiently increase Risk - in addition to its more traditional role of reducing volatility.

In this sense, the paper has two major conclusions. The first is a reinterpretation of the Mutual Fund Theorem. Rather than defining an Efficient Portfolio P^* as a combination of a Risk Free Security and a Risk Portfolio at a given efficient frontier, as traditionally done in Finance textbooks; this paper reinterprets the Efficient Portfolio as a combination of two Efficient Portfolios which, because of different degrees of leverage, lie in different efficient frontiers. The first of these portfolios is the Minimum Variance Portfolio, P_{MIN} , which lies in a frontier where the sum of the portfolio weights of all the securities in the portfolio is exactly equal to the leverage of the portfolio. That is all the value of portfolio P^* is actually invested in this Minimum-Variance Portfolio. The second of these portfolios is the Risk Portfolio, P_{RISK}, in which all tactical allocations are made. The Risk Portfolio lies in a frontier where the sum of all the portfolio weights is exactly equal to zero. That is, every long tactical position in the P_{RISK} portfolio is compensated by short tactical positions - in a manner that all positions end up summing up to zero. It is exactly in this sense that, in the day-to-day jargon of Asset Management, the Minimum Variance Portfolio would be the Strategic Portfolio - where the totality of the resources is invested in long-run allocations - whereas the Risk Portfolio – comprising of the over/underweight tactical allocations that sum up to zero – would be the Tactical Portfolio.

Note that in the traditional Finance literature, the Mutual Fund Theorem is interpreted as stating that every single investor will have exactly the same two portfolios – a Risk Free Security (Minimum Variance Portfolio) and a Risk Portfolio – the only thing differentiating each investor being the amount of resources allocated to each Portfolio; which would depend on the investor´s risk-return preferences. In this paper, on the other hand, we interpret the Mutual Fund Theorem differently. The argument is that every investor will allocate the totality of resources in his/her own Minimum Variance Portfolio and no resources in his/her Risk Portfolio – which will be completely self-financed. Different risk-return preferences will simply result in larger over-weights and under-weights within the Risk Portfolio; all summing up to zero. That is, no positive – or negative - net resources are ever invested in the Risk Portfolio.

In addition, the paper also shows that an increase (decrease) in the leverage of the Efficient Portfolio $P*$ will be completely allocated in the Minimum Variance or Strategic Portfolio, P_{MIN}, leaving the tactical allocations in the Risk Portfolio P_{RISK} completely unaltered. On the other hand, a change in the return/risk preferences will increase the magnitude of the tactical allocations within the Risk Portfolio – leaving the allocation in the Minimum Variance or Strategic Portfolio completely unchanged. This is again in contrast with the traditional interpretation of the Mutual Fund Theorem, in which changes in investor´s return/risk preferences do alter the allocation between the Risk Free Security and the Risk Portfolio – as traditionally defined.

The second major conclusion has to do with the role of Diversification. Diversification has, in fact, two major roles. The first, as traditionally explained in the literature, is related to eliminating idiosyncratic risk; which has to do with the construction of the Minimum Variance Portfolio. In real live, however, minimizing variance eventually leads to fairly concentrated Portfolios[1]. The second role of Diversification, which this paper aims to stress, is its role as an efficient manner to increase the risk of a Portfolio. In fact, what happens is that it suffices for the investor to have one different view on the expected return of a single security in the portfolio, for the Risk or Tactical Portfolio to take positions on all other securities in the Portfolio. In other words, to efficiently capture a higher expected return – and in this sense increase risk – the investor will need to take position on all securities and fully diversify its Risk Portfolio.

The paper is organized as follows. Section II describes the investor´s problem in a mean-variance framework and then derives the first order conditions for an optimal allocation. To allow for the coefficient matrix to be inverted even in the presence of a Risk Free Security, the first order conditions for the investor problem are written in a slightly different manner than in Merton (1972), keeping the Lagrange multipliers on the left-hand side of the equations while moving the Expected Return terms to the right-hand side. This will result in a bordered coefficient matrix that can be inverted even in the presence of a risk free security. That is, contrary to the traditional Finance textbooks, the optimal allocation in the Risk Free Security is determined jointly with the allocation in the risk securities. Section III then states and proves eight properties of an Efficient Allocation while Section IV deals with three properties of the Mean-variance Efficient Portfolio. These two sections, in fact, will define the Minimum Variance Allocations and the Minimum Variance Portfolio as being the equivalents, in Asset Management jargon, to the Strategic Allocation and the Strategic Portfolio – where the totality of the investment is allocated - while the Risk Allocations and the Risk Portfolio would be the equivalents to the Tactical Allocation and Tactical Portfolio – which are completely self-financed. Sections III and IV also prove the properties of these Efficient Allocations and Efficient Portfolios which are, in general, absent from the traditional Finance textbooks. Section V then reinterprets the Mutual Fund Theorem in light of the properties discussed in sections III and IV; while Section VI discusses the new role for Diversification that arises within the Risk – rather than the Minimum Variance - Portfolio. Finally, Section VII concludes.

II. The Investor´s Problem

Consider a portfolio P consisting of N securities with the expected return on the i^{th} security denoted by r_i; the covariance between the returns on the i^{th} and j^{th} securities denoted by σ_{ij}, and the variance of the i^{th} security denoted by $\sigma_{ii} = \sigma_i^2$. Let a_i be the percentage value of portfolio P that is invested in security i. Under these definitions, the return and variance of portfolio P are given by the following expressions:

$$1) R_p = \sum_{i=1}^{N} a_i r_i$$

[1] M. Statman (1987). J. Evan and S. Archer (1968), J. Campbell et all (2001)

2) $\sigma_p^2=\sum_{i=1}^{N} a_i^2\sigma_i^2+\sum_{\square}^{N} \square_N$

The utility of the investor, $U\left(E\left[R_p\right];\sigma_p^2\right)$, is defined over the expected return, $E[R_p]$, and the variance of the portfolio, σ_p^2, - with the first and second order derivatives having the usual signs: $U_1' > 0$; $U_2' < 0$; $U_{11}'' < 0$; $U_{22}'' > 0$. The investor´s problem, is to find a vector of security allocations a_i; $i = 1,\ldots\ldots,N$, that maximizes utility subject to the constraint that the sum of these allocations is equal to the degree of leverage of the portfolio k.

3) $\sum_{i=1}^{N} a_i=k$

Defining λ as the Lagrange multiplier, the Lagrangian for this problem is given by:

$$\Gamma\left(a_i;\lambda\right)=U\left(E\left[R_p\right];\sigma_p^2\right)+\lambda\left[k-\sum_{i=1}^{N} a_i\right]$$

The first order conditions are a system of $N+1$ equations, comprised of equation 3) as well as the following system of N linear equations:

4) $j\neq \underset{N}{i}-\frac{\lambda}{2U_2'}=\frac{-U_1'}{2U_2'}E\left[R_i\right]$ $\qquad i = 1,\ldots\ldots N.$

Notice that the system of N equations in 4) is written with the modified Lagrange multiplier as a right-had side variable; which allows one to write the system of $N + 1$ first order conditions in matrix form as follows:

$$\left[\begin{matrix} \sigma_1^2 & \sigma_{1,2} & \sigma_{1,3} & -\text{¿}-\text{¿}\sigma_{1,N-1} & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \sigma_{2,3} & -\text{¿}-\text{¿}\sigma_{2,N-1} & \sigma_{2,N} \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_3^2 & -\text{¿}-\text{¿}\sigma_{3,N-1} & \sigma_{3,N} \\ -\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}-\text{¿}\sigma_{N-1,N} & \sigma_{N-1,2} & \sigma_{N-1,3} & -\text{¿}-\text{¿}\sigma_{N-1}^2 & \sigma_{N-1,N} \\ \sigma_{N,1} & \sigma_{N,2} & \sigma_{N,3} & -\text{¿}-\text{¿}\sigma_{N,N-1} & \sigma_N^2 \\ 1 & 1 & 1 & -\text{¿}-\text{¿}1 & 1 \end{matrix}\right.$$

Or, using matrix notation,

5) Q . $\boldsymbol{A}$ = $\boldsymbol{R}$

Matrix Q is a $(N+1; N+1)$ coefficient matrix composed by the $(N ; N)$ variance-covariance matrix bordered by the coefficients of the linear constraint in equation 3) as well as the coefficients on the marginal utility adjusted Lagrange multiplier. Notice that, because of the border, matrix Q can still be inverted even in the presence of a Risk Free Security - a security with zero variance and covariance terms. Vector $\boldsymbol{A}$ is a $(N+1 ; 1)$ parameter vector with the elements at the first N rows being the allocation in the N securities of the portfolio, a_i, and the marginal utility adjusted Lagrange multiplier $(-\lambda/2U_2')$ as the element at the $N+1$ row. Vector $\boldsymbol{R}$ is a $(N+1 ; 1)$ right-hand-side vector composed by the N marginal utility adjusted expected returns of the securities in the portfolio; $(-U_1'/2U_2').E[r_i]$, $i = 1, \ldots.., N$, as its first N elements; with the degree of leverage of the portfolio k being the element at the $N+1^{th}$ row. From this point on we will be calling the term $\left(-U_1'/2U_2'\right)$ the risk-return preference parameter.

Adding to the above notation, define four additional $(N+1 ; 1)$ vectors, that will be used in the sections below: vectors $\boldsymbol{Z}$, $\boldsymbol{Z'}$, $\boldsymbol{R'}$, $\boldsymbol{R''}$. Vector $\boldsymbol{Z}$ is composed of N zeros as its $i = 1\ldots\ldots N$ elements and the degree of leverage k as its $N+1^{th}$ element. Vector $\boldsymbol{Z'}$, on the other hand, is composed of N zeros as its $i = 1\ldots\ldots N$ elements and a 1 (one) as its $N + 1th$ element. Vector $\boldsymbol{R'}$, is composed of the N marginal utility adjusted expected return terms, $\left(-U_2'/U_2'\right).E\left[r_i\right]$ as its $i = 1, \ldots., N$ elements, and a zero as its $N+1^{th}$

element. Similarly, vector ***R''***, is composed of the N expected returns, $E[r_i]$ as its $i = 1, \ldots, N$ elements, and a zero as its $N+1^{th}$ element.

Finally, the minors (i, j) of matrix Q will be called $Q_{i,j}$ and, whenever the elements of a certain vector ***V***, for V = ***R, R',Z, Z'*** , are substituted in column i of matrix Q, the resulting matrix will be called Q_i^V.

III. Properties of an Optimal Allocation

One can now state the following properties of an optimal security allocation a_i^*, $i = 1, \ldots N$; where the star superscript represents an optimal,

Property 1:

Every Optimal Allocation a^*_i, $i = 1, \ldots, N$ for any individual security in the portfolio, is a linear combination of two allocations: a Strategic or Minimum Variance Allocation, $a_{i,MIN}$, and a Tactical or Risk Allocation, $a_{i,RISK}$; with the linear coefficient of the Strategic Allocation, $a_{i,MIN}$, being the degree of leverage of the portfolio, k, and the linear coefficient of the Risk Allocation, $a_{i,RISK}$, being the return-risk parameter $U'_1/2U'_2$.

Proof:

From Cramer´s rule we know that the optimal allocations a^*_i are given by:

$$a_i^* = \frac{\mathrm{Det}\left(Q_i^R\right)}{\mathrm{Det}(Q)}, \qquad i = 1, \ldots, N$$

By the properties of determinants, one can add a $N + 1$ column of zeros to the i^{th} column and write the system optimal solutions a^*_i , $i = 1, \ldots, N$ as:

$$a_i^* = \frac{\mathrm{Det}\left(Q_i^Z\right)}{\mathrm{Det}(Q)} + \frac{\mathrm{Det}\left(Q_i^{R'}\right)}{\mathrm{Det}(Q)}, \qquad i = 1, \ldots, N$$

Factoring out the degree of leverage k from $\mathrm{Det}\left(Q_i^Z\right)$ as well as the risk-return preference parameter $\left(-U_1'/2U_2'\right)$ from $\mathrm{Det}\left(Q_i^{R'}\right)$ one can write the above system of equations as follows:

$$a_i^* = k\frac{\mathrm{Det}\left(Q_i^{Z'}\right)}{\mathrm{Det}(Q)} + \frac{U_1'}{2U_2'}\frac{\mathrm{Det}\left(Q_i^{R''}\right)}{\mathrm{Det}(Q)}, \qquad i = 1, \ldots, N$$

Defining:

$$6)\ a_{i,\mathrm{MIN}} = \frac{\mathrm{Det}\left(Q_i^{Z'}\right)}{\mathrm{Det}(Q)}, \qquad i = 1, \ldots, N$$

$$7)\ a_{i,\mathrm{RISK}} = \frac{\mathrm{Det}\left(Q_i^{R''}\right)}{\mathrm{Det}(Q)}, \qquad i = 1, \ldots, N$$

Then, the optimal allocation for security i will be given by the following expression:

$$8)\ a_i^* = \mathrm{k}a_{i,\mathrm{MIN}} + \frac{U_1'}{2U_2'}a_{i,\mathrm{RISK}}, \qquad i = 1, \ldots, N$$

Property 2:

The Strategic or Minimum Variance Allocation, $a_{i,MIN}$, $i = 1, \ldots, N$, for any individual security in the portfolio, is the solution to the problem of minimizing the variance of the portfolio, subject to the constraint that $k = 1$.

Proof:

Consider the problem of minimizing the variance of the portfolio, σ_p^2, subject to the constraint that the leverage of the portfolio is equal to one. Defining $\lambda^{'}$ as the new Lagrange multiplier, the Lagrangian for this problem is given by:

$$\Gamma\left(a_i;\lambda^{'}\right)=\sigma_p^2+\lambda^{'}\left[1-\sum_{i=1}^{N}a_i\right]$$

The first order condition for a minimum is the following system of *N + 1* equations:

9) $\frac{\partial\sigma_{,p}^2}{\partial a_i}-\lambda'=0$

$$1-\sum_{i=1}^{N}a_i=0$$

Which can be represented in matrix notation as:

10) Q . ***A'*** = ***Z***

Where the parameter vector, ***A'***, is exactly equal to vector ***A*** except for the $N + 1^{th}$ row, where we now have the term ' rather than -/2*U'*$_2$.

By Cramer´s rule, the solution to the above system of equations is given by:

$$a_{i,}^{\square}=\frac{\mathrm{Det}\left(Q_i^{Z'}\right)}{\mathrm{Det}\left(Q\right)}$$

Which, as we wanted to prove, is exactly equal to $a_{i,\mathrm{MIN},}^{\square}$.

Property 3:

The Risk Allocation $a_{i,\mathrm{RISK},}$, *i=1.......N for any individual security in the portfolio is the solution to the problem of minimizing the variance of the portfolio subject to the constraint that the leverage of the portfolio ,k, equals zero and the Lagrange multiplier for the constraint* $\overset{-¿=E[R_p]¿}{R}$ *equals one.*

Proof:

Consider the problem of minimizing the variance of the portfolio, σ_p^2, subject to the constraint that the leverage of the portfolio, k, equals to zero and that the expected return on the portfolio $E\left[R_p\right]$ equals to $\overset{-¿¿}{R}$.

Defining $\lambda^{''}$ as the new Lagrande multiplier, the Lagrangian for this problem is given by:

$$\Gamma\left(a_i;\lambda^{''};\gamma\right)=\sigma_p^2+\lambda^{''}\left[0-\sum_{i=1}^{N}a_i\right]+\gamma\,¿$$

The first order condition for a minimum is a system of *N+2* equations given by:

11) $\frac{\partial\sigma_p^2}{\partial a_i}+\lambda''=\gamma E\left[\mathrm{Ri}\right]$ *i = 1, ….., N*

$$0=\sum_{i=1}^{N}a_i$$

$$\overset{-¿-E[R_p]=0¿}{R}$$

Using matrix notation, we can write the system for the first *N+1* equations in 14) and 3´´) as

12) Q . ***A''*** = γ***R''***

Where the parameter vector ***A''*** is exactly equal to the ***A*** vector except for its *N+1 th* term, which is now the new Lagrange multiplier λ'' on the leverage constraint (rather than $-\lambda/2\,U_2'$) and the scalar γ is the Lagrange multiplier on the Return constrain.

By *Cramer's* rule as well as the properties of determinants, one can proceed as before to show that the solution to this variance minimizing problem is given by

$$a_i^{\square} = \gamma \frac{\operatorname{Det}\left(Q_i^{R''}\right)}{\operatorname{Det}(Q)} \qquad i = 1,\dots\dots,N$$

Which, as we wanted to show, is exactly equal to $a_{i,\mathrm{RISK}}$ when γ = 1.

Property 4:

The Strategic, or Minimum Variance Allocation, $a_{i,\mathrm{MIN}}$, *i = 1, ……, N, of any individual security in the portfolio is independent of the expected returns of all securities in the portfolio, depending only on the variance and covariance of these securities*.

Proof:

From equation 6), the Strategic or Minimum Variance Allocation for the *N* securities in the portfolio is given by:

$$a_{i,\mathrm{MIN}} = \frac{\operatorname{Det}\left(Q_i^{Z'}\right)}{\operatorname{Det}Q} \quad \text{for } i = 1\dots\dots\dots N$$

Where matrix *Q*, as described before, is a bordered matrix, in which the (*N ; N*) covariance matrix is boarded by a column and a row of ones and zeros. So, matrix *Q* does not possess any expected returns as its components but only variances and covariance terms. Similarity, the matrix $Q_i^{Z'}$ is the same as matrix *Q* with its i^{th} column substituted by the elements of vector Z', which are zeroes and ones. So it also does not possess any expected return terms as its components; but only variance and covariance terms.

That is, as we wanted to show, all Minimum Variance Allocations $a_{i,\mathrm{MIN}}$ *i = 1, ……, N* are independent of expected returns and depend only on the variances-covariances of the securities in the portfolio.

Property 5:

The Tactical, or Risk Allocation of any individual security in the portfolio, is a linear function of the expected returns of all securities in the portfolio - with the linear coefficient depending only on the variances and covariances of these securities. In addition, for security i, the linear coefficient on the i^{th} *expected return is positive so that an increase (decrease) in the expected return of security i,* $E\left[r_i\right]$ *will increase (decrease) the magnitude of the Tactical Allocation* $a_{i,\mathrm{RISK}}$.

Proof:

From equation 7) the Tactical or Risk Allocation for security *i* is given by the following expression:

$$a_{i,\mathrm{RISK}} = \frac{\operatorname{Det}\left(Q_i^{R''}\right)}{\operatorname{Det}(Q)} \qquad i=1\dots\dots N$$

Where $Q_i^{R''}$ is the same as matrix *Q* with its i^{th} column substituted by the elements of vector *R''*; which are the expected returns $E\left[r_i\right]$; *i = 1,……, N* as elements of the first *N* rows and a zero at its $N+1^{th}$ row term.

By the property of determinants, one can write the determinant of matrix $Q_i^{R"}$ in the following manner:

13) $\mathrm{Det}\left(Q_i^{R"}\right)=E\left[r_i\right]\mathrm{Det}\left(Q_{1,i}\right)+E\left[r_2\right]\mathrm{Det}\left(Q_{2,i}\right)+\ldots.E\left[r_N\right]\mathrm{Det}\left(Q_{N,i}\right)+0.\mathrm{Det}\left(Q_{N+1,i}\right)$

Where the $Q_{j,i}$ terms are the minors *(i, j)* of matrix *Q*. These minors, however, do not depend on the expected returns of the securities in the portfolio $E\left[r_i\right]$, as the i^{th} column is eliminated when calculating the determinant of matrix $Q_i^{R"}$, and depend only on the variance-covariance terms. In addition, matrix *Q* also does not depend on the expected returns, depending only on the variance-covariance terms.

That is, as we wanted to show, all Tactical or Risk allocations, $a_{i,\mathrm{RISK}}$, *i = 1,…..,N*, are linear functions of all the expected returns on the individual securities of the portfolio, with the linear coefficients depending only on the variance-covariance terms.

Finally, let's show that the linear coefficient of the $E\left[r_i\right]$ term, $\mathrm{Det}\left(Q_{\mathrm{ii}}\right)/\mathrm{Det}\left(Q\right)$, is positive – where Q_{ii} is the minor *(i ; i)* of matrix Q. For this, notice that Q_{ii} is exactly matrix *Q* without the i^{th} security. That is, it is exactly as if portfolio *P* had one less security - in this case security *i*. In this sense, for the maximization/minimization problem to be well defined, for both portfolio *P* or for portfolio *P*-without-the-i^{th}-security, the sign of $\mathrm{Det}\left(Q\right)$ would have the same as the sign of $\mathrm{Det}\left(Q_{\mathrm{ii}}\right)$. Thus, as we wanted to show, the linear coefficient of the $E\left[r_i\right]$ term, for the Tactical or Risk Allocation of security *i* , is positive.

Property 6:

The Tactical or Risk Allocation, of any individual security in the portfolio can be written as a linear function of its expected excess returns with respect to a Benchmark Security also in the portfolio - with the linear coefficients depending only on the variances and covariances of the securities in the portfolio. In addition, for security i, the linear coefficient on the i^{th} *expected excess return will be positive, so that an increase (decrease) in the expected excess return of security i with respect to the Benchmark Security, will increase (decrease) the Tactical Allocation on the* i^{th} *security.*

Proof:

Define a Benchmark Security *B* in portfolio *P*.

By the property of determinants, we can subtract row *i = B* in matrix $Q_i^{R"}$ from all other rows *i = 1,.......,N*, for $i \neq B$, while leaving row *N + 1* unaltered. Notice that, by doing this, all elements of the i^{th} column of matrix $Q_i^{R"}$ will become a function of the expected excess returns with respect to the benchmark security *B*, $E\left[r_i - r_B\right]$ - except for the *i = B* row, which will remain $E\left[r_B\right]$. In addition, by subtracting row *i = B* from rows *i = 1, ……, N, i ≠ B*; then, the *N + 1*th column will become a column of zeros, except for the term in the B^{th} row, which will actually be a one. So, if we now calculate the determinant from the B^{th} column we can derive the desired result.

To show that the coefficient on the i^{th} return, $E\left[r_i - r_B\right]$ is positive, we proceed as in Property 5 when we showed that the sign on $\mathrm{Det}\left(Q_{i,i}\right)$ is the same as the sign of $\mathrm{Det}\left(Q\right)$. This is done by simply remembering that, by the properties of the determinants, if one subtracts row *B* from every row *i = 1, ……, N, i ≠ B* of matrix $Q_{i,i}$, it does not alter $\mathrm{Det}\left(Q_{i,i}\right)$. That is, its sign remains unchanged and still equals to the sign of $\mathrm{Det}\left(Q\right)$. In other words, the coefficient $\mathrm{Det}\left(Q_{i,i}\right)/\mathrm{Det}\left(Q\right)$ still has a positive sign, as we wanted to show

Property 7:

If all securities in the portfolio have the same expected return, then the Tactical or Risk Allocation for all securities in the portfolio will be equal to zero and the Optimal Allocation will be equal to the Minimum Variance Allocation.

Proof:

This is immediate from Property 6, which states that the Tactical or Risk Allocation of all securities in the portfolio can be written as a linear function of the excess returns with respect to a benchmark security. That is:

$a_{i,\mathrm{RISK}} = \varphi_{i,1} E[r_1 - r_B] + \varphi_{i,2} E[r_2 - r_B] + \ldots \varphi_{i,1} E[r_i - r_B], \qquad i = 1, \ldots, N$

Thus, if $E[r_i] = E[r_j] = E[r_B]$ for all $i,j = 1, \ldots, N$, then $a_{i,\mathrm{RISK}} = 0$ for all i, as we wanted to show.

Property 8:

If there is, in the portfolio, a security with zero volatility then the Strategic or Minimum Risk Allocation will be equal to one for this security and zero for all other securities in the portfolio.

Proof:

Assume a Risk Free security F with zero volatility. This implies that, in matrix Q, the row and column F, will be a row and column with N zeros elements, $\sigma_{F,j} = \sigma_{i,F} = 0$; $i, j = 1, \ldots, N$, and a one as its N^{th} element; so that $\mathrm{Det}(Q_F^{Z'}) = \mathrm{Det}(Q)$. Thus, from equation 6), $a_{F,\mathrm{MIN}} = 1$.

In addition, for all other securities in the portfolio, $i \neq F$, columns i and F of matrix $Q_F^{Z'}$ will be exactly the same. Thus $\mathrm{Det}(Q_i^{Z'}) = 0$ and, by equation 10), $a_{i,\mathrm{MIN}} = 0$, for all $i \neq F$; as we wanted to prove.

IV. Properties of an Optimal Portfolio

Given the properties of an Efficient Allocation, one can now derive the properties of an Optimal Portfolio.

Property 9:

A Mean-variance Efficient Portfolio is a linear combination of two portfolios: a Strategic or Minimum Variance Portfolio, in which all the value of the portfolio- leveraged or not - is allocated; and a Tactical or Risk Portfolio, in which the magnitude of the Tactical Allocations – that sum to zero – depend on the magnitude of the risk/return preferences of the investor.

Proof:

By Property 1, the optimal allocation is given by:

8) $a_i^{\square} = \mathrm{k}a_{i,\mathrm{MIN}} + \dfrac{U_1^{'}}{U_2^{'}} a_{i,\mathrm{RISK}}, \qquad i = 1, \ldots, N$

One can define a Strategic or Minimum Variance Portfolio P_{MIN}, with $\sum_{i=1}^{N} a_{i,\mathrm{MIN}} = 1$, and with expected return and variance defined by the following equations:

1') $R_{\mathrm{MIN}} = \sum_{i=1}^{N} a_{i,\mathrm{MIN}} r_i$

2') $\sigma_{\mathrm{MIN}}^2 = \sum_{i=1}^{N} a_{i,\mathrm{MIN}}^2 \sigma_i^2 + \sum_{\square}^{N} \square_N$

Similarly one can define a Tactical or Risk Portfolio P_{RISK}, with $\sum_{i=1}^{N} a_{i,\mathrm{RISK}} = 0$, and with expected return and variance defined by the following equations:

1’’) $R_{\mathrm{RISK}} = \sum_{i=1}^{N} a_{i,\mathrm{RISK}} r_i$

2’’) $\sigma_{\mathrm{RISK}}^2 = \sum_{i=1}^{N} a_{i,\mathrm{RISK}}^2 \sigma_i^2 + 2 \sum_{\square}^{N} \square_N$

Thus, from 8), the Return and Variance of the Efficient Portfolio *P**, can be written as functions of portfolios P_{MIN}, and P_{RISK} in the following manner:

14) $R_P^{\square} = \mathrm{kR}_{\mathrm{MIN}} + \left(-U_1' 2 U_2'\right) R_{\mathrm{RISK}}$

This is what we wanted to prove.

In addition, one can calculate the variance of the Efficient Portfolio as a function of the variances and covariances of the Minimum Variance and Risk Portfolios as follows:

15) $\sigma_P^{2^{\square}} = k^2 o_{\mathrm{MIN}}^2 + \left(-U_1' 2 U_2'\right)^2 o_{\mathrm{RISK}}^2 + 2k\left(-U_1' 2 U_2'\right)\sigma_{\mathrm{MIN;RISK}}$

Where $\sigma_{\mathrm{MIN;RISK}}$ is the covariance between the returns of the Minimum Variance and Risk Portfolios.

Property 10:

The Efficient Portfolio, as well as the Minimum Variance Portfolio and the Risk Portfolio, are all efficient frontier portfolios.

Proof:

The problem of finding a given frontier portfolio is a quadratic optimization exercise in which one minimizes the portfolio variance, subject to the constraints that the portfolio weights sum to a constant *k* and the expect return of the portfolio equals a constant $\overset{-¿¿}{R}$.

The Lagrangian for this problem is given by:

$$\Gamma\left(a_i; \lambda''\right) = \sigma_p^2 + \lambda''' \left[k - \sum_{i=1}^{N} a_i\right] + \gamma\,´¿$$

Where λ'’’’ and γ’ are the Lagrange multipliers for the leverage and return constraints in this problem.

The first order conditions comprise a system of *N + 2* equations given by:

$\frac{\partial \sigma_p^2}{\partial a_i} + \lambda''' = \gamma' E[\mathrm{Ri}]$ for *i=1.......N*

$$k = \sum_{i=1}^{N} a_i$$

$$\overset{-¿-E[R_p]=0¿}{R}$$

To derive the efficient frontier, one varies the constraint on the expected return of the portfolio $\overset{-¿¿}{R}$, calculates the effects on the optimal portfolio weights $a_{i,}^{\square}$, *i=1.......N* - given the constraint on the sum of these portfolio weights - and then maps these on the optimal variance and expected return of the portfolio; finally illustrating the frontier in the mean/standard deviation space.

Three things to notice. First, the Efficient Allocations $a_{i,}^{\square}$, *i=1.......N*, of Property 1, are exactly the solution to this problem when $\gamma = \left(-U_1' / 2 U_2'\right)$. Secondly, the Strategic or Minimum Risk Allocations,

$a_{i,\text{MIN}}$, *i=1.......N*, of Property 2, are also the solution to this exact problem, but in this case, when $\gamma = 0$. Finally, the Tactical or Risk Allocations, $a_{i,\text{RISK}}$, *i=1.......N* of Property 3, are is also the solution to this exact problem, but when $\gamma = 1$ and $k = 0$. That is, all three portfolios are frontier portfolios – as we wanted to prove – but, by having a different degree of leverage, *k*, the Tactical or Risk Portfolio lies in a frontier that is different from that of the Efficient and Minimum Variance Portfolios.

Property 11:

An increase (decrease) in the degree of leverage of the Efficient Portfolio will all be allocated in the Strategic or Minimum Variance Portfolio, and none in the Risk or Tactical Portfolio. In addition, it will leave the magnitudes of the Tactical Allocations in the Risk Portfolio unaltered. On the other hand, an increase (decrease) in the return-risk presence of the investor will not alter the allocations either in the Minimum Variance or in the Risk Portfolios. It will, however, increase (decrease) the magnitudes of the Tactical Allocations within the Risk Portfolio.

Proof:

This is evident from the solution for the Efficient Allocation in equation 8)

8) $a_i = \text{k}a_{i,\text{MIN}} + \frac{U_1'}{2U_2'} a_{i,\text{RISK}}, \quad i = 1, \ldots, N$

Thus, an increase (decrease) in *k* will increase (decrease) the allocation in the Minimum Variance Portfolio while leaving the allocation in the Risk Portfolio, as well as the magnitude of the tactical positions within this portfolio, completely unaltered.

On the other hand, a change in the return risk preferences $\left(-U_1'/2U_2'\right)$ will not alter the asset allocation between the Minimum Variance and Risk Portfolios – as $\sum_{i=1}^{N} a_{i,\text{RISK}} = 0$- but will alter the magnitude of the Tactical Allocations within the Risk Portfolio.

V. The Mutual Fund Theorem Reconsidered

Property 9, states that a Mean-variance Efficient Portfolio is a linear combination of two other portfolios; the Strategic or Minimum Variance Portfolio and the Tactical or Risk Portfolio. In the Finance literature this property is traditionally called the Mutual Fund Theorem – a name conveying the notion that the optimal allocation decision for the *N* securities within the portfolio can, in fact, be separated into a simpler allocation decision between two portfolios.

There are a few things to notice about the Mutual Fund Theorem as stated in Property 9. The first one is that the problem defined in Section 1) - that of maximizing an Utility function subject to the constraint that the sum of the shares of all securities in the portfolio equals *k* - can actually be divided into three separated problems: (i) the problem of minimizing the variance of the portfolio subject to the constraint that all invested wealth – leveraged or not - is allocated in the portfolio. (ii) the problem of minimizing the variance of the portfolio subject to an Expected Return constraint as well as the constraint that the portfolio is self-financed – so that all invested wealth allocated to the portfolio is equal to zero (*k = 0*), (iii) and finally the problem of equating the shadow price of the Expected Return constraint in 15), *γ*, to the risk-return preference parameter $\left(-U_1'/U_2'\right)$.

In other words, by Property 9 - and using Asset Management terminology - the solution to the Utility Maximization Problem in Section 1) can be divided in three parts: (i) the definition of a Strategic Portfolio, in which all the invested wealth – leveraged of not – is allocated in the portfolio. This would be the Minimum Variance Portfolio, (ii) the definition of a Tactical Portfolio, in which the sum of the over/underweight positions will all be equal zero. This would be the Risk Portfolio (iii) the calibration of the magnitudes of the Tactical allocations within the Risk Portfolio, according to the investor´s return-risk parameter $\left(-U'_1/U'_2\right)$.

Notice that the above decomposition – or separation – of the Utility Maximization problem in Section 1 is in contrast to the traditional Mutual Fund separation typically stated in Finance textbooks. There, the Minimum Variance Portfolio is actually the Risk Free Security and the Risk Portfolio is comprised of all the other securities in the portfolio. In addition, the adjustment to the risk-return parameter $\left(-U'_1/U'_2\right)$ is actually made by transferring Wealth between the Risk Free Security and the Risk Portfolio – rather than keeping all Wealth allocated in the Minimum Variance Portfolio and merely adjusting the tactical allocations within the Risk Portfolio - as in Proposition 11.

The reason for this difference has to do with how "separation" is carried out in each approach. In the traditional Finance Textbooks, once a Risk Free Security exists, the variance-covariance matrix of the Portfolio cannot be inverted. And the solution to this problem is simply to take the Risk Free Security out of the Portfolio and then carry on with the optimization problem for the remaining *N - 1*securities; which would result in the Risk Portfolio – a portfolio comprised of all securities that are not the Risk Free Security. The Risk Portfolio would then be combined with the Risk Free Security, in accordance to the investor´s risk-return preferences, to form the Efficient Portfolio.

In this paper, however, "separation" is done in a different manner. Even in the presence of a Risk Free Security, matrix *Q* in equation 8) is a bordered variance-covariance matrix and can still be inverted. So the Risk Free Security remains in the Portfolio. Separation is then done b*y* first minimizing the variance of a Portfolio in which all invested wealth is allocated; and then, secondly, by minimizing the variance of a Portfolio where the all Tactical Allocations sum up to zero – but subject to an Expected Return constraint. Thirdly, according to the investor´s Risk Return preferences, the magnitudes of the Tactical Allocations within the Risk Portfolio are then calibrated.

Notice that if there is a Risk Free Security, then as stated in Proposition 8, all the invested wealth would be allocated to it, with the remaining securities being part only of the over/underweight decisions within the Tactical or Risk Portfolio

VI. The Roles of Diversification

According to Property 5, the Risk allocation of a given security in the Portfolio will be a linear function of the Expected Return of all *N* securities in the portfolio – not only of its own Expected Return. This is in contrast to the Minimum Variance allocation, which, according to Property 4, will depend only on the variance-covariances among the *N* securities of the portfolio. Thus, while the Minimum Variance Portfolio can be concentrated – as, for example, in the presence of a Risk Free Security - the Risk Portfolio will continue to be diversified; as will then be the Efficient Portfolio.

Let´s take a closer look at this Diversification argument. Firstly, suppose that the investor expects all securities in the portfolio to have the same exact expected return. Then, by Property 7, there will be zero tactical allocations within the Risk Portfolio and the optimal allocation, for all securities, will be the Minimum Variance Allocation. Secondly, suppose that the investor believes that there is only one single security in the portfolio, say security *i*, with a positive (negative) excess return. In this case, by Property 5, the Risk Allocation in this security will be an overweight (underweight). However, still by Property 5, the Risk Allocation for all other *N - 1* securities in the portfolio will also be different from zero - as they all are a linear functions of the Expected Return on security *i*. In other words, it suffices for the investor to have a view on the excess return of one single security in the portfolio, for the Risk Portfolio to be well diversified. Portfolio Diversification is thus, the efficient solution for taking risk even for the case in which the investor has a view on the Expected Return of one single security.

The above argument clarifies an important practical issue regarding Diversification. While Finance Theory argues in favor of a well-diversified portfolio, as a mean of reducing idiosyncratic risk;

the truth is that a variance minimizing exercise usually results in fairly concentrated portfolios. For instance, a rule of thumb in the practice of Asset Management is that a portfolio comprising of 10 to 15[2] securities would already be sufficiently diversified; with the volatility reduction achieved by further increasing the number of securities in the portfolio being only marginal.

Property 5, however, defines a new role for Diversification in an Efficient Portfolio. Rather than simply reducing portfolio volatility: Diversification is also an important instrument to efficiently increase the risk of a portfolio. In this sense, an over/underweight tactical position in a given security, would be financed by an under/overweight tactical position in all other securities of the portfolio.

VII. Conclusion

By using matrix algebra rather than figures and simple examples, and by allowing for the presence of leverage, this paper derives eleven properties of an Efficient Portfolio. In particular, a Mean-variance Efficient Allocation is a combination of two Efficient Portfolios – each on a different frontier: a Minimum Variance or Strategic Portfolio, where all the invested wealth, leverage or not, is allocated; and a Risk or Tactical Portfolio, where all the tactical over/underweight decisions -that sum up to zero - are implemented.

An increase in the leverage or invested Wealth of the Efficient Portfolio will be all allocated in the Minimum Variance Portfolio, leaving unchanged the over/underweight allocations in the Risk Portfolio. In contrast, a change in the risk-return preference of the investor will only alter the magnitude of the over/underweight allocations within the Risk Portfolio – which will continue to sum up to zero – without altering the allocation in the Minimum Variance Portfolio. This conclusion yields, in a sense, a different perspective to the Mutual Fund Theorem, which is traditionally derived for portfolios with the same leverage and, thus, in the same efficient frontier. In this traditional approach, a change in the return-risk preferences would only change the allocation between the Risk Free Security and the Risk Portfolio, without altering the allocation of securities within the Risk Portfolio. Under the new interpretation, however, the allocation between the portfolios remains unaltered - but the magnitudes of the tactical allocations within the Risk Portfolio are all changed.

This paper also illustrates a different role for diversification – in addition to the traditional role of minimizing portfolio variance. Within the Risk Portfolio, Diversification is simply a manner to efficiently implement a tactical allocation movement and, in fact, if the investor possesses a distinct view on a single excess return, the Risk Portfolio will be fully diversified.

The paper also derives a few other properties of a Mean-variance Efficient Portfolio: (i) The Minimum Variance Allocation for any given security in the portfolio, is solely a function of the variances and covariances and does not depend on the securities expected returns; (ii) The Risk Allocation for any given security is a linear function of the expected returns of all securities in the portfolio, and will depend positively on the expected return of that security itself. (iii) The Risk Allocation for any given security can be written in terms of excess expected returns of all securities in the portfolio, with respect to a benchmark security also in the portfolio; and will depend positively on the expected excess return of that security itself (iv) if the expected return of all securities in the portfolio are the same, the risk allocations will be zero and the optimal allocation will be the Minimum Variance Allocation for all securities in the portfolio (v) if there is a Risk Free Security in the portfolio, the Minimum Variance Allocation for this security will be exactly equal to one and will be zero for all other securities in the portfolio. That is, for the Risk Free Security, the Efficient Allocation will be exactly

[2] See Evans and Archer (1968). More recent work by Statman (1987) argues in favor of 30 to 40 securities for a well-diversified portfolio, whereas the work of Campbell et all (2001) defends that the number of securities in a well-diversified portfolio have increased in the USA.

equal to the Minimum Variance Allocation; while for all other securities in the portfolio it will be exactly equal to the risk allocation. (vi) The Minimum Variance as well as the Risk Portfolio are also Efficient Portfolios; but because of different degrees of leveraging, they lie on different efficient frontiers.

All these properties are usually hidden from traditional Finance textbooks, whose aim is to simplify Modern Portfolio Theory, maybe by too much, in order to attract a larger audience. This, however, has unfortunately resulted in two major misunderstandings that this paper tries to address: a misunderstanding on the true workings of the Mutual Fund Theorem and the dual role of diversification in an efficient Portfolio.

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